THE CO-DEGREE TURÁN PROBLEM ON $K_4^{(3)}$

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ABSTRACT. The tetrahedron $K_4^{(3)}$ is the complete 3–uniform hypergraph on four vertices. We investigate the maximum attainable co-degree parameters in large $K_4^{(3)}$-free 3-graphs. We relate the global edge density $\mu(G) = e(G)/\binom{n}{3}$, the normalized minimum co-degree $\alpha(G) = \delta_2(G)/n$, and the normalized co-degree–squared mass $\beta(G) = \sum_{\{x,y\}} d_G(x,y)^2/[\binom{n}{2}(n-2)^2]$. Furthermore, we revisit the balanced 3-partite construction S_n, introduce a tournament-based construction G_T that yields $\delta_2(G_T) \approx n/2$ while maintaining clique-freeness, and study the induced regularity that dense local configurations force on quadratic statistics of pair degrees. Additionally, we initiate a comparative study of $\pi(K_4^{(3)})$, $\pi_2(K_4^{(3)})$, and the co-degree–squared Turán density $\sigma(K_4^{(3)})$ in the asymptotic regime. We then present open problems, which give a direction to the future study of this field.

1. INTRODUCTION

The turán problems in extremal graph theory stemmed from an original theorem by Pál Turán in the 1940s[11]. Originally Turán asked: *what is the maximum number of edges in an n-vertex graph that does not contain a complete graph K_{r+1}?* The result of this question was the Turán Graph, which established the basis for many more types of these problems.

Inherently, it was seen that k-uniform hypergraphs serve as a natural extesnsion of the Turán Graph.

2. DEFINITIONS AND NOTATION

First we introduce notation that we use throughout the paper.

- For a positive integer n we write $[n]$ for the set $\{1, 2, \ldots, n\}$.
- For a finite set X we write $|X|$ for its cardinality.
- For integers n and k we write $\binom{n}{k}$ for the binomial coefficient and we write $\binom{X}{k}$ for the family of k-element subsets of X.
- For real sequences a_n and b_n we write $a_n = o(b_n)$ to mean $a_n/b_n \to 0$ as $n \to \infty$.
- We write $a_n = O(b_n)$ or $a_n \lesssim b_n$ to mean there exists $C > 0$ with $|a_n| \le C|b_n|$ for all large n.
- We write $a_n \sim b_n$ when $a_n/b_n \to 1$ and $a_n \asymp b_n$ when both $a_n \lesssim b_n$ and $b_n \lesssim a_n$ hold.
- All hypergraphs are finite. Unless otherwise stated n denotes the number of vertices of the host hypergraph.

The definitions below are grouped so that related notions appear together.

Definition 2.1 (Uniform hypergraph)**.** An r-uniform hypergraph or r-graph is a pair $G = (V, E)$ where V is a finite vertex set and $E \subset \binom{V}{r}$ is the set of edges each of size r.

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Definition 2.2 (Vertex and edge counts)**.** For a hypergraph G we write $v(G) = |V(G)|$ for the number of vertices and $e(G) = |E(G)|$ for the number of edges.

Definition 2.3 (Subhypergraph and induced subhypergraph)**.** A subhypergraph H of G is an r-graph with $V(H) \subset V(G)$ and $E(H) \subset E(G)$. The induced subhypergraph $G[U]$ on $U \subset V(G)$ satisfies $E(G[U]) = \{e \in E(G) : e \subset U\}$.

Definition 2.4 (Isomorphism and embedding)**.** Two r-graphs H and G are isomorphic if a bijection $\varphi : V(H) \to V(G)$ maps edges to edges. An embedding of H into G is an injective map from $V(H)$ into $V(G)$ that sends edges of H to edges of G.

Isomorphism and embeddings formalize when a configuration is forbidden or present.

Definition 2.5 (Complete r-graph)**.** The complete r-graph on n vertices is $K_n^{(r)} = (V, \binom{V}{r})$. The tetrahedron $K_4^{(3)}$ denotes the complete 3-uniform hypergraph on four vertices.

Definition 2.6 (Degree of a vertex)**.** For $x \in V(G)$ the vertex degree is $d_G(x) = |\{e \in E(G) : x \in e\}|$.

Definition 2.7 (Maximum and minimum degrees)**.** $\Delta(G) = \max_{x \in V(G)} d_G(x)$. $\delta(G) = \min_{x \in V(G)} d_G(x)$.

Definition 2.8 (Neighborhood and co-degree)**.** For an r-graph G and $S \subset V(G)$ with $|S| = r - 1$ define the neighborhood

$$N_G(S) = \{v \in V(G) \setminus S : S \cup \{v\} \in E(G)\},$$

and the co-degree $d_G(S) = |N_G(S)|$.

Definition 2.9 (Minimum co-degree)**.**

$$\delta_{r-1}(G) = \min\{d_G(S) : S \in \binom{V(G)}{r-1}\}.$$

For $r = 3$ we write $\delta_2(G)$ and for a pair $\{x, y\}$ we also write $c_G(x, y) = d_G(\{x, y\})$.

Vertex degrees and co-degrees are the local parameters that drive both classical Turán problems and co-degree Turán problems. In particular the quantity $\delta_{r-1}(G)$ is the main parameter in co-degree threshold theorems.

Definition 2.10 (Shadow or down-shadow)**.** For a 3-graph G the shadow or down-shadow is

$$\partial G = \{S \in \binom{V(G)}{2} : \exists\, e \in E(G) \text{ with } S \subset e\}.$$

Definition 2.11 (Link graph and induced link notation)**.** For a 3-graph H and vertex $x \in V(H)$ the link graph $L_H(x)$ or $L(x)$ is the graph with vertex set $V(H) \setminus \{x\}$ and edge set $\{ab : abx \in E(H)\}$. For $A \subset V(H)$ write $L_A(x) = L(x)[A]$. For parts A, B define $L_{A,B}(x)$ to be the subgraph of $L(x)$ containing only edges with one endpoint in A and the other in B.

Definition 2.12 (Non-edge subgraph of a link)**.** For parts A, B write $L^c_{A,B}(x)$ for the bipartite graph on $A \cup B$ whose edges are those pairs ab with $a \in A$, $b \in B$ and $abx \notin E(H)$.

Definition 2.13 (Cross-edge counts)**.** For pairwise disjoint sets $A, B, C \subset V(H)$ define

$$e(A,B) = |\{xyz \in E(H) : x,y \in A, z \in B\}| + |\{xyz \in E(H) : x,y \in B, z \in A\}|$$

and

$$e(A,B,C) = |\{abc \in E(H) : a \in A, b \in B, c \in C\}|.$$

The shadow and link notation translate triple counts to graph language. This is used repeatedly.

Definition 2.14 (Weighted pair graph)**.** Given a 3-graph G define the weighted pair graph W_G to be the complete graph on $V(G)$ with integer weight

$$w_G(x,y) = d_G(\{x,y\})$$

assigned to each unordered pair $\{x,y\}$. The underlying support graph of W_G has edge $\{x,y\}$ exactly when $w_G(x,y) > 0$.

Definition 2.15 (Edge weight)**.** For an edge $e = \{x,y,z\} \in E(G)$ define its weight

$$w_G(e) = d_G(x,y) + d_G(x,z) + d_G(y,z).$$

The weighted pair graph and edge weights encode co-degree information in auxiliary graphs. They are essential when converting lower bounds on co-degree into global numerical inequalities and when applying quadratic or ℓ_2 type arguments.

Definition 2.16 (Linearity)**.** A hypergraph G is linear if any two distinct edges intersect in at most one vertex.

Definition 2.17 (Matching and matching number)**.** A matching is a set of pairwise vertex disjoint edges. The matching number $\nu(G)$ is the maximum size of a matching in G.

Definition 2.18 (Vertex cover and cover number)**.** A vertex cover is a vertex set meeting every edge. The vertex cover number $\tau(G)$ is the minimum size of a vertex cover.

Matching and cover parameters often appear in packing and hitting arguments. Linearity frequently simplifies counting because common intersections are controlled.

Definition 2.19 (Complete multipartite r-graph)**.** A complete k-partite r-graph with parts $V_1, \dots, V_k$ contains every r-tuple of vertices that meet each part in at most one vertex.

Definition 2.20 (Balanced multipartition and blow-up)**.** A partition is balanced when part sizes differ by at most one. For an r-graph H and integer $t \geq 1$ the blow-up $H(t)$ is the r-graph obtained by replacing each vertex $v \in V(H)$ by t clones $v_1, \dots, v_t$ and replacing each edge of H by all t^r edges formed by choosing one clone from each vertex of that edge.

Balanced complete multipartite graphs and blow-ups provide a standard family of constructions used for lower bounds. Many extremal examples for both edge Turán and co-degree problems arise as balanced blow-ups of small templates.

Definition 2.21 (Forbidden family and F-free)**.** For a fixed family $\mathcal{F}$ of r-graphs a host graph G is $\mathcal{F}$-free if G contains no subhypergraph isomorphic to any $F \in \mathcal{F}$.

Definition 2.22 (Turán number)**.** For a family $\mathcal{F}$ define the Turán number

$$\mathrm{ex}(n,\mathcal{F}) = \max\{e(G) : |V(G)| = n,\ G \text{ is } \mathcal{F}\text{-free}\}.$$

Definition 2.23 (Turán density)**.**

$$\pi(\mathcal{F}) = \limsup_{n\to\infty} \frac{\mathrm{ex}(n,\mathcal{F})}{\binom{n}{r}}.$$

Definition 2.24 (Minimum co-degree Turán number)**.** For a family $\mathcal{F}$ define

$$\mathrm{ex}_{r-1}(n,\mathcal{F}) = \max\{\delta_{r-1}(G) : |V(G)| = n,\ G \text{ is } \mathcal{F}\text{-free}\}.$$

Definition 2.25 (Minimum co-degree density)**.**

$$\pi_{r-1}(\mathcal{F}) = \limsup_{n\to\infty} \frac{\mathrm{ex}_{r-1}(n,\mathcal{F})}{n}.$$

For $r = 3$ we write $\pi_2(\mathcal{F})$.

The minimum co-degree versions replace edge counts by the minimum co-degree and are the natural comparisons for the co-degree Turán problem. We will compare these quantities to co-degree squared and ℓ_2 notions.

Definition 2.26 (Co-degree squared sum)**.** For a k-graph G define the co-degree squared sum

$$\mathrm{co}_2(G) = \sum_{T\in\binom{V(G)}{k-1}} d_G(T)^2.$$

Definition 2.27 (Co-degree squared extremal function and density)**.** For a family $\mathcal{F}$ define

$$\mathrm{ex}_{\mathrm{co}2}(n,\mathcal{F}) = \max\{\mathrm{co}_2(G) : |V(G)| = n,\ G \text{ is } \mathcal{F}\text{-free}\},$$

and the scaled density

$$\sigma(\mathcal{F}) = \limsup_{n\to\infty} \frac{\mathrm{ex}_{\mathrm{co}2}(n,\mathcal{F})}{\binom{n}{k-1}(n-k+1)^2}.$$

The ℓ_2 perspective is central to recent work and complements classical ℓ_1 Turán research. We will use co_2 to state and prove co-degree squared extremal results and to compare with π and π_{r-1}.

Definition 2.28 (Co-degree threshold)**.** A function $f(n)$ is a co-degree threshold for a fixed r-graph F if every r-graph G on n vertices with $\delta_{r-1}(G) \geq f(n)$ contains F, and lowering $f(n)$ by a growing amount allows F-free constructions. If a constant d works for infinitely many n we call d the co-degree threshold.

Definition 2.29 (Uniform Turán density)**.** A 3-graph $G = (V, E)$ is $(d, \eta, 1)$-dense if for every $U \subset V$,

$$|U^{(3)} \cap E| \geq d\binom{|U|}{3} - \eta|V|^3.$$

The uniform Turán density of H is

$$\pi_u(H) = \sup\{d \in [0,1] : \forall \eta > 0\ \exists \text{ arbitrarily large } H\text{-free } (d,\eta,1)\text{-dense } G\}.$$

Turán Density is typically easier to find then the exact Turán number of a graph, as with most asymptotics.

Definition 2.30 (Named small templates)**.** We adopt the standard names for small 3-graphs used in the literature. For example

- $K_4^{(3)}$ denotes the tetrahedron with edges 123,124,134,234.
- F_5 denotes the 5-vertex 3-graph with edges 123,124,345.
- S_n denotes the complete balanced 3-partite 3-graph on n vertices.
- C_n, B_n, H_6 denote the constructions defined in Section 3 and in the literature.

3. The Turán problem for $K_4^{(3)}$

The 3-uniform Turán problem forbids the tetrahedron

$$K_4^{(3)} = \big(\{1,2,3,4\},\ \{\{i,j,k\} : i,j,k \in [4]\}\big),$$

which is the 3-graph on 4 vertices containing all 4 triples.

We study extremal 3-graphs $G = (V, E)$ on n vertices under this forbidden substructure by viewing three natural density parameters:

- **Edge ("ℓ_1") Density:**

$$\mu(G) = \frac{e(G)}{\binom{n}{3}}, \qquad \pi(K_4^{(3)}) = \limsup_{n\to\infty} \mu(G_{\mathcal{T},n}),$$

  where $G_{\mathcal{T},n}$ denotes any sequence of n-vertex 3-graphs that is $K_4^{(3)}$-free.

- **Minimum Co-Degree Density (Local "ℓ_1"):**

$$\delta_2(G) = \min_{\{x,y\}\subset V(G)} d_G(x,y), \qquad \alpha(G) = \frac{\delta_2(G)}{n}, \qquad \pi_2(K_4^{(3)}) = \limsup_{n\to\infty} \alpha(G_{\mathcal{T},n}).$$

- **Co-degree-Squared ("ℓ_2") Density:**

$$\mathrm{co}_2(G) = \sum_{\{x,y\}\subset V(G)} d_G(x,y)^2, \qquad \beta(G) = \frac{\mathrm{co}_2(G)}{\binom{n}{2}(n-2)^2}, \qquad \sigma(K_4^{(3)}) = \limsup_{n\to\infty} \beta(G_{\mathcal{T},n}).$$

Even before proving explicit inequalities, the three parameters are conceptually related by the following themes:

(1) Forbidding $K_4^{(3)}$ imposes strong structure on link graphs $L(x)$: each $L(x)$ must be triangle-free, otherwise a triangle in a link together with its apex forms a banned tetrahedron. Thus, a local condition (triangle-freeness of every link) propagates global density restrictions.

(2) Two complementary natural constructions are known in the literature and reproduced later:

- A balanced tripartite transversal 3-graph, which lifts Mantel's 2-part equilibrium[8] to 3 parts, maximizing $\mu(G)$ without creating $K_4^{(3)}$ because it eliminates all co-degree inside any part
- A tournament-based opposition construction, which generates $\delta_2(G) \approx n/2$ while still avoiding $K_4^{(3)}$, since no single vertex can be the smallest in more than 3 incident transversal edges without forcing forced orientation conflicts in the link.

These demonstrate that best lower bounds for π and π_2 arise from completely different templates.

(3) If a 3-graph is close to edge-extremal, then most pairs $\{x, y\}$ must satisfy $d_G(x, y)$ near some constant multiple of n, otherwise the total number of edges would drop. When pair degrees are forced into a narrow interval, the quadratic statistic $\mathrm{co}_2(G)$ becomes large. Hence σ is not independent of π: *denser 3-graphs implicitly yield bigger ℓ_2 mass simply because pair degrees cannot fluctuate too widely without lowering edge count.

We also recall two standard facts that justify why all later bounds fall in the range they do:

- Every triangle-free graph on m vertices has at most $\frac{1}{4}m^2 + O(m)$ edges. Since each link $L_G(x)$ is triangle-free and has $n-1$ vertices, this gives the rough local constraint

$$e(L_G(x)) \leq \tfrac{1}{4}(n-1)^2 + O(n).$$

- Averaging $e(L_G(x))$ over all $x \in V$ double counts pairs inside triples:

$$\sum_x e(L_G(x)) = 3e(G).$$

If all links could simultaneously achieve Mantel extremality, this heuristic calculation would predict only

$$\mu(G) \approx \frac{1/4 \cdot n^2}{2(n/3)} = \frac{2}{3},$$

which becomes the trivial ceiling for edge density in 3-uniform clique Turán problems. The best modern computational bound is only slightly smaller (< 0.562), and the classical conjectured value from S_n is $5/9 \approx 0.5555$. Thus even prior to formal proof, the literature constrains all three densities to:

$$0.5 \lesssim \pi_2(K_4^{(3)}) \leq 1, \qquad 0.5555 \lesssim \pi(K_4^{(3)}) \leq 1, \qquad 0 \leq \sigma(K_4^{(3)}) \leq 1.$$

We will now provide the construction for explicit bounds that make this a much more narrow interval.

4. Upper Bound Techniques

In this section we discuss several mechanisms for bounding from above the extremal quantities attached to $K_4^{(3)}$: the ordinary Turán density $\pi(K_4^{(3)})$, the co-degree Turán density $\pi_2(K_4^{(3)})$ and the co-degree-squared density $\sigma(K_4^{(3)})$. The guiding theme is that local parameters such as $\delta_2(G)$ and the ℓ_2-quantity $\mathrm{co}_2(G)$ exert strong control over the global edge density of a $K_4^{(3)}$-free 3-graph G.

Throughout this section F denotes a fixed 3-graph and G is an F-free 3-graph on n vertices. For G we write

$$\mu(G) = \frac{e(G)}{\binom{n}{3}}, \qquad \alpha(G) = \frac{\delta_2(G)}{n}, \qquad \beta(G) = \frac{\mathrm{co}_2(G)}{\binom{n}{2}(n-2)^2},$$

so that $\mu(G), \alpha(G), \beta(G) \in [0, 1]$ for all G.

4.1. **Local to global inequalities.** We begin with the basic relations between co-degrees, the co-degree-squared sum and the number of edges. These are all consequences of double counting and a single application of Cauchy–Schwarz.

Lemma 4.1. *Let G be a 3-graph on n vertices. Then*

$$\sum_{\{x,y\}\in\binom{V(G)}{2}} d_G(x,y) = 3\,e(G),$$

$$\sum_{\{x,y\}\in\binom{V(G)}{2}} d_G(x,y)^2 \geq \binom{n}{2}\delta_2(G)^2,$$

and

$$9\,e(G)^2 \leq \binom{n}{2}\mathrm{co}_2(G).$$

Proof. Every edge $\{x,y,z\}$ contributes 1 to each of $d_G(x,y)$, $d_G(x,z)$ and $d_G(y,z)$, so in total it contributes 3 to the sum

$$\sum_{\{x,y\}} d_G(x,y),$$

and the first identity follows by double counting.

For the second inequality, note that $d_G(x,y) \geq \delta_2(G)$ for every unordered pair $\{x,y\}$, hence

$$\mathrm{co}_2(G) = \sum_{\{x,y\}} d_G(x,y)^2 \geq \sum_{\{x,y\}} \delta_2(G)^2 = \binom{n}{2}\delta_2(G)^2.$$

For the third inequality we apply Cauchy–Schwarz to the family of numbers $a_{xy} = d_G(x,y)$ indexed by unordered pairs $\{x,y\}$:

$$\Big(\sum_{\{x,y\}} d_G(x,y)\Big)^2 \leq \binom{n}{2}\sum_{\{x,y\}} d_G(x,y)^2 = \binom{n}{2}\mathrm{co}_2(G).$$

Using the first identity to rewrite the left-hand side as $9e(G)^2$ gives

$$9e(G)^2 \leq \binom{n}{2}\mathrm{co}_2(G),$$

as claimed. ∎

In terms of the normalized quantities $\mu(G)$, $\alpha(G)$ and $\beta(G)$ we obtain the following.

Corollary 4.2. *Let G be a 3-graph on $n \geq 3$ vertices. Then*

$$\alpha(G) \leq \mu(G)\,\frac{n-2}{n},$$

and

$$\alpha(G)^2 \leq \beta(G)\,\frac{(n-2)^2}{n^2}, \qquad \mu(G)^2 \leq \beta(G).$$

Proof. From Lemma 4.1 we have

$$3e(G) = \sum_{\{x,y\}} d_G(x,y) \geq \binom{n}{2}\delta_2(G) = \binom{n}{2}\alpha(G)n,$$

so

$$e(G) \geq \frac{\alpha(G)n}{3}\binom{n}{2}.$$

Dividing by $\binom{n}{3} = n(n-1)(n-2)/6$ yields

$$\mu(G) = \frac{e(G)}{\binom{n}{3}} \geq \frac{\alpha(G)n}{3} \cdot \frac{\binom{n}{2}}{\binom{n}{3}} = \alpha(G)\,\frac{n}{n-2},$$

which rearranges to $\alpha(G) \leq \mu(G)\,(n-2)/n$.

For the relation between $\alpha(G)$ and $\beta(G)$, Lemma 4.1 gives

$$\mathrm{co}_2(G) \geq \binom{n}{2}\,\delta_2(G)^2 = \binom{n}{2}\,\alpha(G)^2 n^2,$$

hence

$$\beta(G) = \frac{\mathrm{co}_2(G)}{\binom{n}{2}(n-2)^2} \geq \alpha(G)^2\,\frac{n^2}{(n-2)^2}.$$

Rearranging yields $\alpha(G)^2 \leq \beta(G)\,(n-2)^2/n^2$.

Finally, combining Lemma 4.1 with the definition of $\beta(G)$,

$$9e(G)^2 \leq \binom{n}{2}\,\mathrm{co}_2(G) = \binom{n}{2}^2 (n-2)^2\,\beta(G).$$

Taking square roots and dividing by $\binom{n}{3}$ gives

$$\mu(G) = \frac{e(G)}{\binom{n}{3}} \leq \frac{\binom{n}{2}(n-2)}{3\binom{n}{3}}\sqrt{\beta(G)} = \sqrt{\beta(G)},$$

since $\binom{n}{2}(n-2)/(3\binom{n}{3}) = 1$. ∎

In particular, for large n the three normalized parameters satisfy

$$\alpha(G) \leq \mu(G) + o(1), \alpha(G)^2 \leq \beta(G) + o(1), \text{ and } \mu(G)^2 \leq \beta(G).$$

Passing to limits along extremal sequences gives a clean hierarchy between the three densities introduced in Section 2.

Proposition 4.3. *For every fixed 3-graph F one has*

$$\pi_2(F) \leq \pi(F) \leq \sqrt{\sigma(F)}.$$

Proof. Let (G_n) be a sequence of F-free 3-graphs with $|V(G_n)| = n$ and

$$\limsup_{n\to\infty} \frac{\delta_2(G_n)}{n} = \pi_2(F).$$

Applying Corollary 4.2 to each G_n gives

$$\frac{\delta_2(G_n)}{n} \leq \frac{e(G_n)}{\binom{n}{3}} \cdot \frac{n-2}{n} \leq \frac{e(G_n)}{\binom{n}{3}},$$

for all large n. Taking lim sup and using the definitions of $\pi_2(F)$ and $\pi(F)$ yields $\pi_2(F) \leq \pi(F)$.

Similarly, let (H_n) be a sequence of F-free 3-graphs with $|V(H_n)| = n$ and

$$\limsup_{n\to\infty} \frac{\mathrm{co}_2(H_n)}{\binom{n}{2}(n-2)^2} = \sigma(F).$$

Then Corollary 4.2 implies

$$\frac{e(H_n)}{\binom{n}{3}} \le \sqrt{\frac{\mathrm{co}_2(H_n)}{\binom{n}{2}(n-2)^2}},$$

for all n. Taking lim sup on both sides gives $\pi(F) \le \sqrt{\sigma(F)}$. ∎

For $F = K_4^{(3)}$ this shows that any effective upper bound on $\pi(K_4^{(3)})$ or on $\sigma(K_4^{(3)})$ automatically yields an upper bound on the co-degree Turán density $\pi_2(K_4^{(3)})$ as well.

4.2. **Edge-density bounds for $K_4^{(3)}$.** The most basic upper bounds for clique Turán problems come from averaging and symmetrization. For r-uniform hypergraphs, a simple argument of de Caen[5] shows that

$$\pi\big(K_{r+1}^{(r)}\big) \le 1 - \frac{1}{r}, \qquad r \ge 2,$$

so in particular

$$\pi(K_4^{(3)}) \le \frac{2}{3}.$$

This already improves the trivial bound 1, but it is far from the conjectured value 5/9 suggested by the tripartite construction S_n in Section 4.

A substantial improvement comes from working with weighted hypergraphs and their Lagrangians. Let G be a 3-graph on vertex set V. For a probability vector $(x_v)_{v\in V}$ with $x_v \ge 0$ and $\sum_{v\in V} x_v = 1$, define the Lagrangian

$$\lambda(G) = \sum_{\{a,b,c\}\in E(G)} x_a x_b x_c.$$

If G is F-free, then any blow-up $G(t)$ is still F-free, and a standard calculation shows

$$\lim_{t\to\infty} \mu\big(G(t)\big) = \max_{(x_v)} \lambda(G),$$

where the maximum is over all probability distributions on $V(G)$. It follows that

$$\pi(F) \le \sup\{\lambda(G) : G \text{ is } F\text{-free}\}.$$

Using this approach, Chung and Lu[3], building on earlier work of de Caen[5] and Giraud[7], showed that for $F = K_4^{(3)}$ we have

$$\sup\{\lambda(G) : G \text{ is } K_4^{(3)}\text{-free}\} \le \frac{3+\sqrt{17}}{12} \approx 0.5936.$$

In particular,

$$\pi(K_4^{(3)}) \le \frac{3+\sqrt{17}}{12}.$$

Their argument works as follows. We fix an arbitrary $K_4^{(3)}$-free 3-graph G and an optimal weight vector (x_v) and then extracts a large 4-partite subhypergraph of G that captures almost all of $\lambda(G)$. From there, it counts $K_4^{(3)}$-configurations inside that 4-partite structure in terms of the part sums $X_i = \sum_{v\in V_i} x_v$. Finally it optimizes a quartic polynomial in the X_i under the constraints $X_i \ge 0$ and $\sum_i X_i = 1$. We omit the technicallies here.

The outcome is that no $K_4^{(3)}$-free 3-graph can have Lagrangian larger than $(3+\sqrt{17})/12$, and therefore the same constant bounds the ordinary Turán density from above.

4.3. **Flag algebras and the best current upper bound.** The strongest known upper bounds on $\pi(K_4^{(3)})$ come from Razborov's flag algebra method[9]. In this subsection we outline the formal setup and indicate how one obtains an explicit inequality of the form

$$\mu(G) \le 0.56167 + o(1)$$

for every $K_4^{(3)}$-free 3-graph G.

Homomorphism and induced densities. Fix an integer $m \ge 4$ and let $\mathcal{H}_m$ be the family of all $K_4^{(3)}$-free 3-graphs on vertex set $[m]$, considered up to isomorphism. For $H \in \mathcal{H}_m$ and a large $K_4^{(3)}$-free 3-graph G on n vertices define the induced density

$$p(H;G) = \mathbb{P}\big(G[U] \cong H\big),$$

where $U \subset V(G)$ is a uniformly random m-subset. Then

$$\sum_{H \in \mathcal{H}_m} p(H;G) = 1.$$

Passing to a subsequence if necessary, any sequence of $K_4^{(3)}$-free 3-graphs (G_n) with $|V(G_n)| \to \infty$ has well-defined limits

$$p_H = \lim_{n\to\infty} p(H;G_n), \qquad H \in \mathcal{H}_m.$$

The edge density of G can be expressed in terms of these induced densities. Indeed, the expected number of edges in $G[U]$ is

$$\mathbb{E}\big[e(G[U])\big] = \sum_{H \in \mathcal{H}_m} e(H)\, p(H;G),$$

while by double counting we also have

$$\mathbb{E}\big[e(G[U])\big] = \mu(G)\binom{m}{3} + O\Big(\frac{1}{n}\Big).$$

Thus

$$\mu(G) = \sum_{H \in \mathcal{H}_m} \mu(H)\, p(H;G) + O\Big(\frac{1}{n}\Big),$$

where $\mu(H) = e(H)/\binom{m}{3}$. Passing to the limit along (G_n) gives

$$\lim_{n\to\infty} \mu(G_n) = \sum_{H \in \mathcal{H}_m} \mu(H)\, p_H.$$

In addition to the trivial constraint $\sum_H p_H = 1$, the vector $(p_H)_{H\in\mathcal{H}_m}$ satisfies many further linear relations coming from consistency of overlaps between different induced subgraphs. The flag algebra method systematically exploits these relations together with positive semidefinite quadratic forms.

Types, flags and a basic inequality. A *type* σ is a small labelled 3-graph. We take σ to be a single labelled vertex. A *σ-flag* is a 3-graph F together with an injective map $\theta : V(\sigma) \to V(F)$ that identifies the labelled vertices. Isomorphic flags are identified.

Fix a type σ and an integer $\ell > |\sigma|$. Let $\mathcal{F}_\ell^\sigma$ be the set of all σ-flags on ℓ vertices, up to isomorphism. Given a large 3-graph G and a random injective labeling $\theta : V(\sigma) \to V(G)$, we can define for $F \in \mathcal{F}_\ell^\sigma$ the *flag density*

$$p(F; G, \theta) = \mathbb{P}\big(G[W] \cong F \text{ as a labelled 3-graph}\big),$$

where W is a random ℓ-vertex superset of $\theta(V(\sigma))$.

Now let $Q = (q_{F,F'})_{F,F' \in \mathcal{F}_\ell^\sigma}$ be a real symmetric positive semidefinite matrix. For a fixed G and random θ consider the random vector

$$X(G, \theta) = (p(F; G, \theta))_{F \in \mathcal{F}_\ell^\sigma} \in \mathbb{R}^{\mathcal{F}_\ell^\sigma},$$

and the quadratic form

$$\Phi_Q(G, \theta) = X(G, \theta)^\top Q\, X(G, \theta).$$

Since Q is positive semidefinite we have $\Phi_Q(G, \theta) \geq 0$ for every θ. Taking expectation over θ gives

$$\mathbb{E}_\theta\big[\Phi_Q(G, \theta)\big] \geq 0.$$

The key point is that $\mathbb{E}_\theta[\Phi_Q(G, \theta)]$ can be rewritten as a linear combination of the induced densities $p(H; G)$ for $H \in \mathcal{H}_m$ with m depending only on ℓ and the type σ. More precisely, for each pair (F, F') and each $H \in \mathcal{H}_m$ one can compute the probability that a random labelling θ and random extensions of size ℓ around θ give flags F and F' sitting inside an induced copy of H. Summing over all F, F' with weights $q_{F,F'}$ yields coefficients $c_H(Q) \in \mathbb{R}$ such that

$$\mathbb{E}_\theta\big[\Phi_Q(G, \theta)\big] = \sum_{H \in \mathcal{H}_m} c_H(Q)\, p(H; G).$$

Thus we obtain the inequality

$$\sum_{H \in \mathcal{H}_m} c_H(Q)\, p(H; G) \geq 0.$$

Passing to the limit along a convergent sequence (G_n) gives

$$\sum_{H \in \mathcal{H}_m} c_H(Q)\, p_H \geq 0.$$

In words: every positive semidefinite matrix Q on flags produces one explicit linear inequality satisfied by the limiting induced density vector (p_H).

The semidefinite program and the bound for $K_4^{(3)}$. The flag algebra method for $K_4^{(3)}$ proceeds as follows.

- Fix m. in Razborov's computation he takes $m = 7$. And enumerate the $K_4^{(3)}$-free 3-graphs $\mathcal{H}_m$.
- Choose a small collection of types σ and corresponding flag sizes ℓ.
- For each (σ, ℓ), consider a positive semidefinite matrix variable $Q^{(\sigma,\ell)}$ indexed by $\mathcal{F}_\ell^\sigma$, and let $c_H^{(\sigma,\ell)}(Q^{(\sigma,\ell)})$ be the corresponding coefficients as above.

The edge density of a large $K_4^{(3)}$-free 3-graph satisfies

$$\mu(G) = \sum_{H \in \mathcal{H}_m} \mu(H)\, p(H;G) + O\Big(\frac{1}{n}\Big),$$

while for each (σ, ℓ) the positivity of $\mathbb{E}_\theta[\Phi_{Q^{(\sigma,\ell)}}(G,\theta)]$ yields

$$\sum_{H \in \mathcal{H}_m} c_H^{(\sigma,\ell)}(Q^{(\sigma,\ell)})\, p(H;G) \geq 0.$$

By taking a suitable linear combination of these inequalities together with the trivial constraints $\sum_H p(H;G) = 1$ and $p(H;G) \geq 0$, one aims to obtain an inequality of the form

$$\mu(G) \leq \vartheta + \sum_{H \in \mathcal{H}_m} d_H\, p(H;G) + O\Big(\frac{1}{n}\Big),$$

where the coefficients d_H arise from the matrices $Q^{(\sigma,\ell)}$ and are such that

$$\sum_{H \in \mathcal{H}_m} d_H\, p(H;G) \geq 0$$

for every $K_4^{(3)}$-free G. If this is achieved with some real number ϑ, then letting $n \to \infty$ along an arbitrary extremal sequence (G_n) gives

$$\pi\big(K_4^{(3)}\big) \leq \vartheta.$$

This becomes a semidefinite program (SDP). The unknowns are the entries of the matrices $Q^{(\sigma,\ell)}$. We maximize the constant ϑ, and the constraints encode that each $Q^{(\sigma,\ell)}$ is positive semidefinite and that the resulting coefficients d_H are nonnegative. Solving this SDP numerically and then rounding the solution gives matrices $Q^{(\sigma,\ell)}$ and a constant ϑ. It also makes sure all the inequalities hold

Razborov's computation for $K_4^{(3)}$[10] yields matrices $Q^{(\sigma,\ell)}$ and a constant

$$\vartheta = 0.56167\ldots$$

for which the above scheme works. More precisely, one obtains a finite family of inequalities

$$\sum_{H \in \mathcal{H}_m} c_H^{(j)}\, p(H;G) \geq 0, \qquad j = 1, \ldots, J,$$

such that

$$\mu(G) \leq 0.56167 + \sum_{j=1}^{J} \sum_{H \in \mathcal{H}_m} c_H^{(j)}\, p(H;G) + O\Big(\frac{1}{n}\Big),$$

and each inner sum is nonnegative for every $K_4^{(3)}$-free G. This implies the following theorem.

Theorem 4.4. *Every $K_4^{(3)}$-free 3-graph G on n vertices satisfies*

$$\mu(G) \leq 0.56167 + o_n(1).$$

Consequently

$$\pi\big(K_4^{(3)}\big) \leq 0.56167.$$

The proof is done once the matrices $Q^{(\sigma,\ell)}$ and the resulting coefficients $c_H^{(j)}$ has been verified. This verification itself is done by computer arithmetic. We will not need the explicit form of the certificate in what follows, only the numerical upper bound on $\pi(K_4^{(3)})$.

Combining Theorem 4.4 with the lower bound from the tripartite construction S_n in Section 4 shows that

$$\frac{5}{9} \le \pi\big(K_4^{(3)}\big) \le 0.56167,$$

so the true value of $\pi(K_4^{(3)})$ lies in a narrow interval of width less than 0.01 around the conjectured value $5/9$.

4.4. **Known bounds for the co-degree Turán problem.** The co-degree Turán density of a 3-graph F is

$$\pi_2(F) = \limsup_{n\to\infty} \frac{\mathrm{ex}_2(n,F)}{n},$$

where $\mathrm{ex}_2(n,F)$ is the maximum possible $\delta_2(G)$ over all F-free 3-graphs G on n vertices.

For $F = K_4^{(3)}$ the exact value of $\pi_2(F)$ is unknown. On the upper-bound side, Proposition 4.3 and Theorem 4.4 give

$$\pi_2\big(K_4^{(3)}\big) \le \pi\big(K_4^{(3)}\big) \le 0.56167.$$

On the lower-bound side, constructions of Czygrinow and Nagle[4] (later generalized by Falgas-Ravry[6]) show that for every $s \ge 4$ there exist $K_s^{(3)}$-free 3-graphs on n vertices whose minimum co-degree is asymptotically $\big(1 - \frac{1}{s-2}\big)n$. Specializing to $s = 4$ yields

$$\pi_2\big(K_4^{(3)}\big) \ge \frac{1}{2}.$$

Czygrinow and Nagle conjectured that this lower bound is in fact best possible, that is,

$$\pi_2\big(K_4^{(3)}\big) = \frac{1}{2}.$$

This would match the behavior observed for several other small 3-graphs where the co-degree density is known exactly and is strictly smaller than the ordinary Turán density. At present, however, no nontrivial upper bound below $\pi(K_4^{(3)})$ is known for $\pi_2(K_4^{(3)})$, and combining the inequalities above we only have

$$\frac{1}{2} \le \pi_2\big(K_4^{(3)}\big) \le 0.56167.$$

4.5. **Upper bounds from the ℓ_2-extremal problem.** The co-degree-squared extremal problem for $K_4^{(3)}$ asks for the asymptotics of

$$\mathrm{ex}_{\mathrm{co2}}(n, K_4^{(3)}) = \max\big\{\mathrm{co}_2(G) : |V(G)| = n,\ G \text{ is } K_4^{(3)}\text{-free}\big\}.$$

In the notation of Section 2 this is captured by the density

$$\sigma\big(K_4^{(3)}\big) = \limsup_{n\to\infty} \frac{\mathrm{ex}_{\mathrm{co2}}(n, K_4^{(3)})}{\binom{n}{2}(n-2)^2}.$$

Balogh, Clemen and Lidický[2] showed that for every fixed 3-graph F the scaled limit defining $\sigma(F)$ actually exists and, in particular, determined $\sigma(K_4^{(3)})$ via a flag algebra computation. Moreover, they proved that every $K_4^{(3)}$-free 3-graph G with $\mathrm{co}_2(G)$ close to $\mathrm{ex}_{\mathrm{co2}}(n, K_4^{(3)})$

must be structurally close to a balanced blow-up of one of the Turán-type candidate constructions for $K_4^{(3)}$.

Combined with Proposition 4.3, their result yields the inequalities

$$\pi\big(K_4^{(3)}\big) \le \sqrt{\sigma\big(K_4^{(3)}\big)}, \qquad \pi_2\big(K_4^{(3)}\big) \le \sqrt{\sigma\big(K_4^{(3)}\big)}.$$

Numerically the resulting upper bound on $\pi(K_4^{(3)})$ is currently weaker than the flag algebra bound of Theorem 4.4, but the ℓ_2-setting has a significant structural advantage: near-extremal values of $\mathrm{co}_2(G)$ force the co-degrees $d_G(x,y)$ to be highly regular across almost all pairs $\{x,y\}$.

In particular, if one could show that every $K_4^{(3)}$-free 3-graph with minimum co-degree

$$\delta_2(G) \ge (c - o(1))n$$

must already satisfy

$$\mathrm{co}_2(G) \ge \big(\sigma\big(K_4^{(3)}\big) - o(1)\big)\binom{n}{2}(n-2)^2,$$

then the structural stability theory developed for the ℓ_2-extremal problem would automatically transfer to the co-degree Turán problem. This is one of the motivations behind the stability question formulated in Problem 7.1 in Section 7.

5. A Construction For Lower Bounds

In this section we discuss constructions giving lower bounds for the three extremal densities attached to $K_4^{(3)}$ introduced earlier:

$$\pi\big(K_4^{(3)}\big), \qquad \pi_2\big(K_4^{(3)}\big), \qquad \sigma\big(K_4^{(3)}\big).$$

The guiding theme is that suitably structured $K_4^{(3)}$-free 3-graphs can simultaneously exhibit large global edge density, large minimum co-degree, and large co-degree-squared mass. The local–global inequalities from Section 4, in particular Corollary 4.2 and Proposition 4.3, then allow us to transfer information between these three parameters.

We begin with the classical Turán (edge-density) construction, then treat the minimum co-degree via a tournament-based construction, and finally deduce lower bounds for the co-degree-squared density using both constructions together with the inequalities from Section 4.

5.1. **The tripartite construction for $\pi(K_4^{(3)})$.** The complete balanced 3-partite 3-graph S_n provides the best known lower bound construction for the ordinary Turán density. Its edge count satisfies

$$e(S_n) = \big(\tfrac{5}{9} + o(1)\big)\binom{n}{3}, \tag{5.1}$$

which immediately implies the lower bound

$$\pi(K_4^{(3)}) \ge \tfrac{5}{9}. \tag{5.2}$$

This construction was first given by Turán and remains conjectured to be extremal[11]. All known upper bounds are only slightly above this value; see Section 4 for the best current numerical estimates for $\pi(K_4^{(3)})$.

We now describe S_n in detail and verify that it is $K_4^{(3)}$-free and has asymptotic edge density 5/9.

Definition of S_n and $K_4^{(3)}$-freeness. Let n be large, and partition the vertex set $V = \{1, \dots, n\}$ into three parts

$$V = V_1 \cup V_2 \cup V_3, \qquad |V_i| = \frac{n}{3} + o(n),$$

where the sizes differ by at most one. Define a 3-uniform hypergraph S_n on vertex set V by declaring that a triple $\{x, y, z\} \subseteq V$ is an edge of S_n iff the three vertices lie in three distinct parts:

$$E(S_n) = \{\{x, y, z\} : x \in V_1,\ y \in V_2,\ z \in V_3\} \quad \text{and all permutations of } (1, 2, 3).$$

Thus S_n is the complete 3-partite 3-graph in the *transversal* sense, with the part sizes as balanced as possible.

The structure of S_n mirrors the classical Turán graph for K_3 but lifted to the 3-uniform setting: edges correspond to choosing one vertex from each color class.

To see that S_n contains no copy of $K_4^{(3)}$, consider any set of four vertices a, b, c, d. By the pigeonhole principle, since the partition has only three parts, at least two of the four vertices lie in the same part. WLOG, suppose $a, b \in V_1$.

Any triple containing both a and b lies entirely within at most two parts, and therefore violates the defining rule for edges of S_n. In particular, the triple $\{a, b, c\}$ is not an edge of S_n for any choice of c.

But in $K_4^{(3)}$, every 3-subset of the four vertices must be an edge. Since at least one of the triples among $\{a, b, c, d\}$ is absent in S_n, no quadruple of vertices induces $K_4^{(3)}$. Therefore

$$S_n \text{ is } K_4^{(3)}\text{-free.}$$

Counting the edges of S_n. By construction, every edge corresponds to a choice of one vertex from each part, hence

$$e(S_n) = |V_1|\,|V_2|\,|V_3|.$$

When the partition is balanced, we have

$$|V_i| = \frac{n}{3} + o(n), \qquad i = 1, 2, 3.$$

Therefore,

$$e(S_n) = \left(\frac{n}{3} + o(n)\right)\left(\frac{n}{3} + o(n)\right)\left(\frac{n}{3} + o(n)\right) = \left(\frac{1}{27} + o(1)\right) n^3.$$

To express this in normalized form, recall that the total number of triples in an n-vertex 3-graph is

$$\binom{n}{3} = \frac{n^3}{6} + o(n^3).$$

Combining these two estimates, we obtain

$$\frac{e(S_n)}{\binom{n}{3}} = \frac{\frac{1}{27}n^3 + o(n^3)}{\frac{1}{6}n^3 + o(n^3)} = \frac{1/27}{1/6} + o(1) = \frac{5}{9} + o(1).$$

Thus the edge density of S_n converges to $5/9$ as $n \to \infty$. Since each S_n is $K_4^{(3)}$-free and contains $\left(\frac{5}{9} + o(1)\right)\binom{n}{3}$ edges, we obtain

$$\pi(K_4^{(3)}) \geq \lim_{n\to\infty} \frac{e(S_n)}{\binom{n}{3}} = \frac{5}{9}.$$

No construction with higher asymptotic density is currently known, and it is also unknown whether this is the exact value of $\pi(K_4^{(3)})$.

5.2. **A tournament construction for $\pi_2(K_4^{(3)})$.** We now turn to the co-degree Turán density

$$\pi_2(F) = \limsup_{n\to\infty} \frac{\mathrm{ex}_2(n, F)}{n},$$

where $\mathrm{ex}_2(n, F)$ denotes the largest possible minimum co-degree $\delta_2(G)$ over all F-free 3-graphs G on n vertices.

For $F = K_4^{(3)}$, Czygrinow and Nagle gave a construction (later generalized by Falgas-Ravry) that shows

$$\pi_2\big(K_4^{(3)}\big) \geq \frac{1}{2}.$$

We give here a natural version of their construction based on tournaments and verify the lower bound.

From a tournament to a 3-graph.

Definition 5.1 (Tournament construction)**.** Let $n \in \mathbb{N}$ and let T be a tournament on vertex set $[n]$, i.e. an orientation of the edges of K_n in which every pair of vertices is joined by exactly one directed edge. For $i, j \in [n]$ with $i \neq j$, write $i \to j$ if the edge is oriented from i to j.

Define a 3-graph G_T on vertex set $[n]$ as follows: for each triple $i < j < k$, we declare

$$\{i, j, k\} \in E(G_T) \quad \Longleftrightarrow \quad (i \to j \text{ and } k \to i) \text{ or } (j \to i \text{ and } i \to k).$$

In words: $\{i, j, k\}$ is an edge if, from the point of view of the smallest vertex i, the orientations of the two edges (i, j) and (i, k) are opposite.

Thus the rule deciding whether a given triple is an edge depends only on the tournament edges incident with its smallest vertex.

$K_4^{(3)}$-freeness of the tournament construction.

Lemma 5.2. *For every tournament T on $[n]$, the 3-graph G_T is $K_4^{(3)}$-free.*

Proof sketch. Suppose for a contradiction that some 4-set $\{a, b, c, d\}$ with $a < b < c < d$ spans a copy of $K_4^{(3)}$ in G_T. Then all four triples

$$\{a, b, c\}, \quad \{a, b, d\}, \quad \{a, c, d\}, \quad \{b, c, d\}$$

must be edges.

By Definition 5.1, the status of each triple is decided by the orientations of the two tournament edges incident with its smallest vertex. For example, $\{a, b, c\}$ and $\{a, b, d\}$ are decided only by how (a, b), (a, c) and (a, d) are oriented, while $\{b, c, d\}$ depends only on the orientations from b to c and d.

We can now check that no choice of orientations around a and b can satisfy the edge condition for all four triples simultaneously. Whenever $\{a,b,c\}$ and $\{a,b,d\}$ satisfy the rule at a, this forces a pattern that prevents at least one of $\{a,c,d\}$ or $\{b,c,d\}$ from satisfying the rule at its smallest vertex.

Hence there is no copy of $K_4^{(3)}$ in G_T. ■

Typical co-degrees in the tournament construction. We next show that for a *random* tournament, almost all pairs have co-degree close to $n/2$, and in fact the minimum co-degree is asymptotically $n/2$ with high probability.

Lemma 5.3. *Let T be a uniformly random tournament on $[n]$, and let G_T be the associated 3-graph. Then with high probability*

$$\delta_2(G_T) \geq \Big(\tfrac{1}{2} - o(1)\Big)n.$$

Proof. Fix an unordered pair $\{u,v\} \subseteq [n]$. For each third vertex $w \in [n]\setminus\{u,v\}$, consider the triple $\{u,v,w\}$. Depending on the relative order of u,v,w, one of them is the smallest; say i. Then the rule deciding whether $\{u,v,w\}$ is an edge of G_T depends only on the orientations of the two edges incident with i among $\{u,v,w\}$.

In a random tournament, each edge orientation is an independent fair coin flip. For a fixed triple $\{u,v,w\}$, the two relevant edges are therefore independent and each direction is equally likely. It follows directly from the definition that for a fixed triple, the event that $\{u,v,w\}$ is an edge of G_T has probability exactly $1/2$.

Moreover, for different choices of w these events are independent. Consequently, for each fixed pair $\{u,v\}$, the random co-degree $d_{G_T}(u,v)$ has distribution

$$d_{G_T}(u,v) \sim \mathrm{Bin}(n-2, 1/2),$$

with mean $(n-2)/2$ and variance $(n-2)/4$.

By a standard Chernoff bound, for every fixed $\varepsilon > 0$ we have

$$\mathbb{P}\Big(\big|d_{G_T}(u,v) - \tfrac{1}{2}(n-2)\big| > \varepsilon n\Big) \leq 2e^{-c\varepsilon^2 n}$$

for some absolute constant $c > 0$. Taking a union bound over the $\binom{n}{2}$ pairs $\{u,v\}$ and choosing $\varepsilon = o(1)$ slowly tending to zero, we see that with probability tending to 1 as $n \to \infty$,

$$d_{G_T}(u,v) \geq \Big(\tfrac{1}{2} - o(1)\Big)n \qquad \text{for every pair } \{u,v\}.$$

That is, $\delta_2(G_T) \geq (\frac{1}{2} - o(1))n$ with high probability. ■

Combining Lemmas 5.2 and 5.3 yields the desired lower bound for $\pi_2(K_4^{(3)})$.

Proposition 5.4. *There exists a sequence of $K_4^{(3)}$-free 3-graphs (G_n) with $|V(G_n)| = n$ such that*

$$\frac{\delta_2(G_n)}{n} \geq \frac{1}{2} - o(1).$$

In particular

$$\pi_2\big(K_4^{(3)}\big) \geq \frac{1}{2}.$$

Proof. For each n choose a tournament T_n on $[n]$ for which Lemma 5.3 holds (one exists with positive probability), and let $G_n := G_{T_n}$. Then G_n is $K_4^{(3)}$-free by Lemma 5.2, and satisfies $\delta_2(G_n) \geq (\frac{1}{2} - o(1))n$ by Lemma 5.3. Taking the limit of the suprenum in the definition of $\pi_2(K_4^{(3)})$ gives the claimed inequality. ∎

5.3. **Lower bounds for the co-degree-squared density $\sigma(K_4^{(3)})$.** Finally we consider the co-degree-squared extremal problem for $K_4^{(3)}$,

$$\sigma\big(K_4^{(3)}\big) = \limsup_{n\to\infty} \frac{\mathrm{ex}_{\mathrm{co2}}(n, K_4^{(3)})}{\binom{n}{2}(n-2)^2},$$

where

$$\mathrm{co}_2(G) = \sum_{\{x,y\}} d_G(x,y)^2$$

is the co-degree-squared sum and $\mathrm{ex}_{\mathrm{co2}}(n,F)$ maximizes this quantity over all n-vertex F-free 3-graphs.

A general inequality relating $\pi(F)$ and $\sigma(F)$. Recall from Corollary 4.2 that for every 3-graph G,

$$\mu(G)^2 \leq \beta(G),$$

where

$$\mu(G) = \frac{e(G)}{\binom{n}{3}}, \qquad \beta(G) = \frac{\mathrm{co}_2(G)}{\binom{n}{2}(n-2)^2}.$$

Passing to extremal limits along F-free sequences gives the first inequality in Proposition 4.3,

$$\pi(F) \leq \sqrt{\sigma(F)}.$$

Rewriting, we obtain the general lower bound

$$\sigma(F) \geq \pi(F)^2 \qquad \text{for every fixed 3-graph } F. \tag{5.3}$$

Applying the inequality to $K_4^{(3)}$. For $F = K_4^{(3)}$, combining (5.3) with the lower bound (5.2) from the tripartite construction S_n yields:

Corollary 5.5. *For the complete 3-graph $K_4^{(3)}$ we have*

$$\sigma\big(K_4^{(3)}\big) \geq \pi\big(K_4^{(3)}\big)^2 \geq \left(\frac{5}{9}\right)^2 = \frac{25}{81}.$$

In particular, the same sequence (S_n) that provides the best known lower bound for the ordinary Turán density also yields, via the inequality $\mu(G)^2 \leq \beta(G)$, a nontrivial lower bound on the normalized co-degree-squared density.

An alternative lower bound from the tournament construction. The tournament-based construction from Subsection 5.2 gives a complementary lower bound for $\sigma(K_4^{(3)})$ arising from the fact that it has very large minimum co-degree.

Lemma 5.6. *Let $G_n := G_{T_n}$ be as in Proposition 5.4. Then*

$$\liminf_{n\to\infty} \frac{\mathrm{co}_2(G_n)}{\binom{n}{2}(n-2)^2} \geq \frac{1}{4}.$$

Consequently

$$\sigma\big(K_4^{(3)}\big) \geq \frac{1}{4}.$$

Proof. By Proposition 5.4 we have

$$\delta_2(G_n) \geq \Big(\frac{1}{2} - o(1)\Big)n.$$

Hence every pair $\{x, y\}$ satisfies

$$d_{G_n}(x, y) \geq \Big(\frac{1}{2} - o(1)\Big)n,$$

and so

$$\mathrm{co}_2(G_n) = \sum_{\{x,y\}} d_{G_n}(x, y)^2 \geq \binom{n}{2}\Big(\frac{1}{2} - o(1)\Big)^2 n^2 = \Big(\frac{1}{4} - o(1)\Big)\binom{n}{2}(n-2)^2 \cdot \frac{n^2}{(n-2)^2}.$$

As $n \to \infty$ we have $\frac{n^2}{(n-2)^2} \to 1$, so

$$\frac{\mathrm{co}_2(G_n)}{\binom{n}{2}(n-2)^2} \geq \frac{1}{4} - o(1),$$

and taking the lim inf gives the claim. ∎

Lemma 5.6 is weaker than Corollary 5.5 (since $1/4 < 25/81$), but it illustrates how a construction optimized for the *minimum* co-degree automatically forces a substantial amount of ℓ_2-mass in the co-degree distribution.

6. Asymptotic Results

In this section we pass to the limit $n \to \infty$ and view $K_4^{(3)}$–free 3–graphs through the three normalized parameters

$$\mu(G) = \frac{e(G)}{\binom{n}{3}}, \alpha(G) = \frac{\delta_2(G)}{n}, \text{ and } \beta(G) = \frac{\mathrm{co}_2(G)}{\binom{n}{2}(n-2)^2}.$$

For a sequence (G_n) of $K_4^{(3)}$–free 3–graphs with $|V(G_n)| \to \infty$ we are interested in the limit points of the triples

$$\big(\mu(G_n), \alpha(G_n), \beta(G_n)\big) \in [0, 1]^3.$$

The extremal quantities $\pi(K_4^{(3)})$, $\pi_2(K_4^{(3)})$ and $\sigma(K_4^{(3)})$ measure the largest possible limiting values of μ, α and β separately. The goal here is to record what is known about these limits and to explain a simple identity which ties β to μ and the dispersion of pair co-degrees.

6.1. **The variance identity for co-degrees.** Fix a 3–graph G on n vertices and write

$$\mathcal{P} = \binom{V(G)}{2}$$

for the set of unordered pairs. For each $\{x, y\} \in \mathcal{P}$ let $d_{xy} = d_G(x, y)$. Define the average co-degree

$$\overline{d} = \frac{1}{|\mathcal{P}|} \sum_{\{x,y\} \in \mathcal{P}} d_{xy}$$

and the variance

$$\mathrm{Var}(d_G) = \frac{1}{|\mathcal{P}|} \sum_{\{x,y\} \in \mathcal{P}} \big(d_{xy} - \overline{d}\big)^2.$$

Lemma 6.1. *For every 3–graph G on $n \geq 3$ vertices one has*

$$\beta(G) = \mu(G)^2 + \frac{\mathrm{Var}(d_G)}{(n-2)^2}.$$

In particular $\beta(G) \geq \mu(G)^2$, with equality if and only if all pair co-degrees are equal.

Proof. By Lemma 4.1,

$$\sum_{\{x,y\} \in \mathcal{P}} d_{xy} = 3e(G).$$

Since $|\mathcal{P}| = \binom{n}{2}$ this gives

$$\overline{d} = \frac{3e(G)}{\binom{n}{2}}.$$

Using the identity $\binom{n}{3} = \frac{n-2}{3}\binom{n}{2}$ and the definition $\mu(G) = e(G)/\binom{n}{3}$ we obtain

$$e(G) = \mu(G)\binom{n}{3} = \mu(G)\,\frac{n-2}{3}\binom{n}{2},$$

hence

$$\overline{d} = \frac{3e(G)}{\binom{n}{2}} = \mu(G)(n-2).$$

On the other hand,

$$\mathrm{co}_2(G) = \sum_{\{x,y\} \in \mathcal{P}} d_{xy}^2 = |\mathcal{P}|\big(\overline{d}^{\,2} + \mathrm{Var}(d_G)\big),$$

since $\frac{1}{|\mathcal{P}|}\sum d_{xy}^2 = \overline{d}^{\,2} + \mathrm{Var}(d_G)$ by definition of variance. Dividing by $\binom{n}{2}(n-2)^2$ gives

$$\beta(G) = \frac{\mathrm{co}_2(G)}{\binom{n}{2}(n-2)^2} = \frac{|\mathcal{P}|}{\binom{n}{2}} \cdot \frac{\overline{d}^{\,2} + \mathrm{Var}(d_G)}{(n-2)^2} = \frac{\overline{d}^{\,2}}{(n-2)^2} + \frac{\mathrm{Var}(d_G)}{(n-2)^2}.$$

Substituting $\overline{d} = \mu(G)(n-2)$ yields the claimed formula

$$\beta(G) = \mu(G)^2 + \frac{\mathrm{Var}(d_G)}{(n-2)^2}.$$

Since $\mathrm{Var}(d_G) \geq 0$, we have $\beta(G) \geq \mu(G)^2$ with equality only when $\mathrm{Var}(d_G) = 0$, i.e. when all d_{xy} are equal. ∎

Combining Lemma 6.1 with Corollary 4.2 and Proposition 4.3 yields the following picture.

Corollary 6.2. *Let (G_n) be any sequence of 3–graphs with $|V(G_n)| \to \infty$, and suppose that*

$$\mu(G_n) \to \mu_\infty, \qquad \alpha(G_n) \to \alpha_\infty, \qquad \beta(G_n) \to \beta_\infty.$$

Then

$$0 \le \alpha_\infty \le \mu_\infty \le 1, \qquad \mu_\infty^2 \le \beta_\infty \le 1, \qquad \alpha_\infty^2 \le \beta_\infty,$$

and

$$\beta_\infty - \mu_\infty^2 = \lim_{n\to\infty} \frac{\mathrm{Var}(d_{G_n})}{(n-2)^2}.$$

In particular the gap $\beta_\infty - \mu_\infty^2$ is exactly the normalized asymptotic variance of the pair co-degrees.

For a fixed forbidden configuration F, every accumulation point $(\mu_\infty, \alpha_\infty, \beta_\infty)$ arising from an F–free sequence must satisfy these inequalities, and the extremal constants $\pi(F)$, $\pi_2(F)$ and $\sigma(F)$ are simply the largest possible values of μ_∞, α_∞ and β_∞ among such triples.

6.2. **Asymptotic constants for the tetrahedron.** We now specialize to $F = K_4^{(3)}$ and summarize the best currently known asymptotic information on the three densities

$$\pi(K_4^{(3)}), \qquad \pi_2(K_4^{(3)}), \qquad \sigma(K_4^{(3)}).$$

Theorem 6.3. *For the tetrahedron $K_4^{(3)}$ the following hold.*

(1) *The ordinary Turán density satisfies*

$$\frac{5}{9} \le \pi\big(K_4^{(3)}\big) \le 0.5615.$$

(2) *The minimum co-degree Turán density satisfies*

$$\frac{1}{2} \le \pi_2\big(K_4^{(3)}\big) \le 0.529.$$

(3) *The co-degree–squared Turán density is known exactly:*

$$\sigma\big(K_4^{(3)}\big) = \frac{1}{3}.$$

Discussion of the bounds. The lower bound $\pi(K_4^{(3)}) \ge 5/9$ is obtained by a 3–partite Turán construction C_n attributed to Turán, Brown, Kostochka and Fon-der-Flaass; C_n is $K_4^{(3)}$–free and satisfies $\mu(C_n) \to 5/9$ as $n \to \infty$[11]. The best current upper bound $\pi(K_4^{(3)}) \le 0.5615$ is due to Baber[1], who refines Razborov's flag algebra calculation using partially defined and vertex–colored 3–graphs.

For the co-degree Turán density, Czygrinow and Nagle constructed $K_4^{(3)}$–free 3–graphs with minimum co-degree asymptotic to $n/2$, which gives $\pi_2(K_4^{(3)}) \ge 1/2$[4]. Their construction can be formulated in terms of a random tournament on $[n]$ and the 3–graph G_T defined in Definition 5.1. Balogh, Clemen and Lidický applied flag algebras in the co-degree setting and proved the upper bound $\pi_2(K_4^{(3)}) \le 0.529$[2].

Finally, the ℓ_2 version of the tetrahedron problem was solved by Balogh, Clemen and Lidický, who showed that

$$\sigma\big(K_4^{(3)}\big) = \frac{1}{3}$$

with the lower bound achieved by the 3–partite construction C_n and the matching upper bound obtained via flag algebras[2]. ∎

Combining Theorem 6.3 with Proposition 4.3 and Lemma 6.1 we obtain the asymptotic inequalities

$$\frac{1}{2} \le \pi_2\big(K_4^{(3)}\big) \le \pi\big(K_4^{(3)}\big) \le \sqrt{\sigma\big(K_4^{(3)}\big)} = \frac{1}{\sqrt{3}},$$

and

$$\pi\big(K_4^{(3)}\big)^2 \le \sigma\big(K_4^{(3)}\big) = \frac{1}{3}.$$

Numerically this places $\pi(K_4^{(3)})$ in a window of width less than 0.01 around Turán's conjectured value $5/9$, and $\pi_2(K_4^{(3)})$ in a window of width less than 0.03 around the natural lower bound $1/2$.

6.3. **Asymptotic profiles of the main constructions.** The three constructions discussed earlier give accumulation points of the triple (μ, α, β) and show the trade offs between maximizing edge density, minimum co-degree and ℓ_2–mass of co-degrees.

The 3*–partite construction.* Let C_n be the 3–graph with vertex partition $V_1 \cup V_2 \cup V_3$, where the parts are as equal as possible, and edge set

$$\begin{aligned} E(C_n) = &\{abc : a \in V_1,\ b \in V_2,\ c \in V_3\} \\ &\cup \{abc : a, b \in V_1,\ c \in V_2\} \\ &\cup \{abc : a, b \in V_2,\ c \in V_3\} \\ &\cup \{abc : a, b \in V_3,\ c \in V_1\}. \end{aligned}$$

This is the construction used in Theorem 6.3. It is $K_4^{(3)}$–free and satisfies

$$\mu(C_n) \to \frac{5}{9}, \text{ and } \beta(C_n) \to \frac{1}{3}$$

as $n \to \infty$.

A straightforward counting of co-degrees shows that, for n divisible by 3,

- pairs inside a part have co-degree approximately $n/3$;
- pairs across different parts have co-degree approximately $2n/3$;
- the minimum co-degree satisfies $\alpha(C_n) \to 1/3$.

Thus the asymptotic triple associated with this construction is

$$(\mu, \alpha, \beta) \approx \Big(\frac{5}{9}, \frac{1}{3}, \frac{1}{3}\Big).$$

In particular C_n simultaneously witnesses the lower bounds $\pi(K_4^{(3)}) \ge 5/9$ and $\sigma(K_4^{(3)}) \ge 1/3$.

The tournament construction. Let T be a uniformly random tournament on $[n]$ and G_T the associated 3–graph from Definition 5.1. For each triple $\{i, j, k\}$ with $i < j < k$, the orientations of the two edges incident with i are independent fair coin flips, and exactly half of the outcomes make $\{i, j, k\}$ an edge of G_T. Hence

$$\mathbb{E}\big[e(G_T)\big] = \frac{1}{2}\binom{n}{3}, \text{ where } \mu(G_T) \to \frac{1}{2}$$

with high probability.

For each fixed pair $\{u, v\}$, the random co-degree $d_{G_T}(u, v)$ is distributed as $\mathrm{Bin}(n-2, 1/2)$, so $d_{G_T}(u, v)$ has mean $(n-2)/2$ and variance $(n-2)/4$. Thus

$$\alpha(G_T) \geq \Big(\frac{1}{2} - o(1)\Big)n$$

with high probability (Lemma 5.3), so $\alpha(G_T) \to 1/2$. Moreover

$$\mathbb{E}\big[d_{G_T}(u, v)^2\big] = \Big(\frac{n-2}{2}\Big)^2 + \frac{n-2}{4},$$

and averaging over pairs shows that

$$\beta(G_T) \to \frac{1}{4}.$$

Equivalently, $\mu(G_T)^2 \to \beta(G_T)$ and the normalized variance term in Lemma 6.1 tends to 0. Summarizing, the tournament construction yields the asymptotic triple

$$\big(\mu, \alpha, \beta\big) \approx \Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}\Big).$$

This realization shows that it is possible to have asymptotically optimal minimum co-degree (up to the constant $1/2$) while keeping both the edge density and the co-degree–squared density strictly below their conjectural maxima.

The extremal picture for $\sigma(K_4^{(3)})$. For the co-degree–squared problem, Balogh, Clemen and Lidický[2] established a stability theory showing that every $K_4^{(3)}$–free 3–graph G with $\beta(G)$ very close to $\sigma(K_4^{(3)}) = 1/3$ must be structurally close to a blow-up of the 3–partite construction C_n. In particular, any sequence (G_n) with

$$\beta(G_n) \to \frac{1}{3}$$

must have edge density $\mu(G_n)$ tending to at most $\sqrt{1/3}$ and at least $5/9$, and its co-degree distribution must have normalized variance

$$\frac{\mathrm{Var}(d_{G_n})}{(n-2)^2} = \beta(G_n) - \mu(G_n)^2$$

tending to a number in the interval

$$\Big[\frac{1}{3} - \pi\big(K_4^{(3)}\big)^2,\ \frac{1}{3} - \Big(\frac{5}{9}\Big)^2\Big] = \Big[0,\ \frac{2}{81}\Big],$$

with the upper endpoint attained by C_n. Thus in the ℓ_2–extremal regime almost all pair co-degrees must be highly regular.

7. Open Problems

The following open problems are, to our knowledge, unsolved as of 2025. Here are the different things that will happen in each.

Problem 7.1 (Stability Problem for the Co-Degree Turán Threshold of $K_4^{(3)}$)**.** Determine the structural stability of $K_4^{(3)}$–free 3–uniform hypergraphs under large minimum co-degree.

More precisely, prove (or disprove) that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if G is a $K_4^{(3)}$–free 3–graph on n vertices satisfying

$$\delta_2(G) \geq \big(\tfrac{5}{9} - \delta\big)\, n,$$

then there exists a tripartition $V(G) = A \cup B \cup C$ with

$$\left|E(G) \triangle E(S_n)\right| \leq \varepsilon n^3,$$

where S_n is the balanced complete 3–partite 3–graph on n vertices.

In other words, any $K_4^{(3)}$–free 3–graph whose minimum co-degree is close to the conjectured extremal value must be structurally close to the tripartite construction S_n.

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